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Scalar and spinning particles in a plane-wave field

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Abstract
We study the quantization problem of relativistic scalar and spinning particles interacting with a radiation electromagnetic field by using the path integral and the external source method. The spin degrees of freedom are described in terms of Grassmann variables and the Feynman kernel is obtained through functional integration on both Bose and Fermi variables. We provide rigorous proof that the Feynman amplitudes are only determined by the classical contribution and we explicitly evaluate the propagators.

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1. Introduction

Many years ago we studied the quantum-mechanical interaction of a relativistic material point (scalar particle) with an external radiation field using the Feynman path integral method [1]. The eigenfunctions of this problem had been known for a long time [2], and a beautiful method of evaluating the Green functions had been introduced by Fock and Schwinger through the solutions of the Heisenberg quantum equations of motion in the proper time representation [3, 4]. A direct calculation in terms of path integral had remained unexplored probably due to mathematical difficulties. However, looking at the problem in a semiclassical way, in [1] we found a canonical transformation which made it possible to evaluate in a closed form the Feynman kernel for a scalar particle interacting with an external electromagnetic wave field and to obtain an exact expression for the propagator of the theory.

Later on a lot of interest was devoted to theories involving anticommuting (Grassmann) variables. This was mainly raised by supersymmetries [5], but very soon the relevance of anticommuting variables in many other fields was realized [6–9]. In particular, it was shown that Grassmann variables are suitable tools for giving a *classical* description of spin [6–8] and internal degrees of freedom of elementary particles [10]. These dynamical theories described by Lagrangians involving ordinary c-numbers and anticommuting numbers (Grassmann variables) were called *pseudoclassical theories* and the general approach has been defined as *pseudomechanics*, due to the special nature of the variables occurring in the problem.

Having obtained a pseudoclassical description of many interesting physical systems it was very natural to investigate the quantization of such systems by path integrals, performed on both the ordinary and the Grassmann variables (for the general theory of integration on Grassmann algebras see [11]). This programme had already been developed in some cases; for instance it was shown that the Wilson loop could be reconstructed as a path integral on Grassmann variables describing the colour degrees of freedom [10, 12]. Other physical systems, such as non-relativistic spinning particles interacting with constant electric and magnetic field and relativistic spinning particles in external crossed static electromagnetic wave field were studied. In each case, the result was obtained in a quick and straightforward way, by solving the classical equations of motion both for position and Grassmann variables. Just a bit of caution had to be taken for systems involving an odd number of Grassmann variables, like for the non-relativistic and for the massive relativistic spinning particle: it was shown in [13, 14] how to extend the path integral techniques in such circumstances, by separating from the total phase space a coupled one-dimensional system and studying the general solution for the latter. For a different approach to the path integral quantization for a non-relativistic and relativistic spinning particle by using BRST-invariant path integral, see [15].

A further powerful instrument that can be brought to bear to the present context is the well-known external source approach (in our context real and Grassmann sources). Indeed, the aim of this paper is to provide a rigorous determination of the Feynman propagator for a charged relativistic particle, both scalar and spinning, interacting with an arbitrary external electromagnetic wave field by using path integral and external source formalism.

One can ask about the intrinsic interest of this approach apart from the pedagogical one. We would suggest that it is at least twofold. In the first place the development of new techniques to solve old problems usually increases their comprehension. Secondly, new techniques can produce new approximation methods to apply to new problems. In this particular case we would like to emphasize the use of these methods in statistical mechanics, where an impressive number of old problems were solved in a very fast way by using path integrals over Grassmann variables [16].

This paper is organized as follows. In section 2 we present the results for the relativistic material point in this new framework: they are certainly simpler, due to the simpler underlying physical model (scalar particle). Next, in section 3, we switch to the spinning particle and prove that this approach works well in this case too and evaluate the physical Feynman kernel. In section 4 we express the physical kernel on a spinor basis, obtaining a more transparent expression for the Feynman propagator.

2. The path integral for the scalar particle

The Lagrangian describing the dynamics of a relativistic scalar particle interacting with an external electromagnetic field is given by

$$\mathcal{L}(x,\dot{x}) = -m\sqrt{\dot{x}^2} - e(\dot{x}\cdot A). \tag{1}$$

This Lagrangian is singular, giving rise to the constraint

$$\chi = [(p-eA)^2 - m^2] \tag{2}$$

and to a vanishing canonical Hamiltonian. According to Dirac [17] we define an extended Hamiltonian

$$\mathcal{H}_E(x,p,\alpha_1) = \alpha_1[(p-eA)^2 - m^2] \tag{3}$$

and require the usual form of canonical Poisson brackets

$$\{x^\mu, p^\nu\} = -\eta^{\mu\nu} \qquad \eta^{\mu\nu} = (+, -, -, -). \tag{4}$$

The Lagrange multiplier α_1 for the constraint χ must be chosen to be negative in order to have a definite positive kinetic part. The extended Lagrangian corresponding to $\mathcal{H}_E$ is

$$\mathcal{L}_E(x, \dot{x}, \alpha_1) = \dot{x}^2/(4\alpha_1) - e(\dot{x} \cdot A) + \alpha_1 m^2. \tag{5}$$

We assume that the electromagnetic potential A^μ describes the field of an external plane wave and therefore is of the form

$$A^\mu = \epsilon_1^\mu f(\phi) \tag{6}$$

where $\phi = (k \cdot x)$, $k^\mu = \frac{1}{\sqrt{2}}(1, 0, 0, 1)$ is the propagation vector and ϵ_1^μ a transverse real polarization vector. We find it useful to introduce a second transverse vector ϵ_2^μ and the conjugate light-like vector $\tilde{k}^\mu = \frac{1}{\sqrt{2}}(1, 0, 0, -1)$: they satisfy the orthonormality relations

$$k^2 = \tilde{k}^2 = (k \cdot \epsilon_1) = (\tilde{k} \cdot \epsilon_1) = (k \cdot \epsilon_2) = (\tilde{k} \cdot \epsilon_2) = 0 \tag{7}$$

$$(\tilde{k} \cdot k) = -\epsilon_1^2 = -\epsilon_2^2 = 1. \tag{8}$$

We shall derive the amplitude for the propagation of the particle interacting with the external field, using the path integral technique as developed by Feynman [18]. If $x_i = x(\tau_i)$ is the initial point, and $x_f = x(\tau_f)$ is the final one ($\tau_i < \tau_f$), we get

$$K_{fi} = K(x_f, \tau_f, x_i, \tau_i) = c \int_{(x_i,\tau_i)}^{(x_f,\tau_f)} \mathcal{D}[x(\tau)] \exp\left(\mathrm{i}\int_{\tau_i}^{\tau_f} \mathrm{d}\tau\, \mathcal{L}_E(x, \dot{x}, \alpha_1)\right) \tag{9}$$

where the constant c is determined by the condition

$$\lim_{\tau_f \to \tau_i} K(x_f, \tau_f, x_i, \tau_i) = \delta^4(x_f - x_i) \tag{10}$$

and the physical propagator is obtained by the integral

$$K_{\text{phys}} = \int_{-\infty}^{0} \mathrm{d}(\alpha_1 \Delta\tau) K_{fi} \tag{11}$$

where $\Delta\tau = \tau_f - \tau_i$. The calculation is done by introducing the shift $x^\mu(\tau) = x_c^\mu(\tau) + y^\mu(\tau)$ where $x_c^\mu(\tau)$ is the classical path which satisfies the equation of motion

$$\ddot{x}_c^\mu(\tau) = -2\alpha_1 e F^\mu_\nu(x_c(\tau))\dot{x}_c^\nu(\tau) \tag{12}$$

and by integrating over the deviation $y^\mu(\tau)$ from the classical path $x_c^\mu(\tau)$ with the boundary conditions $y^\mu(\tau_f) = y^\mu(\tau_i) = 0$. In equation (12), F^μ_ν is the electromagnetic tensor of the potential A^μ. A series expansion in terms of y^μ then gives

$$K_{fi} = \mathrm{e}^{\mathrm{i}S_c} \int_{(0,\tau_i)}^{(0,\tau_f)} \mathcal{D}[y(\tau)] \exp\left\{\mathrm{i}\int_{\tau_i}^{\tau_f} \mathrm{d}\tau \left[\frac{1}{4\alpha_1}\dot{y}^2 - e(\epsilon_1 \cdot \dot{x}) \sum_{n=2}^{\infty} \frac{f^{(n)}(\phi)}{n!}(k \cdot y)^n \right.\right.$$
$$\left.\left. - e(\epsilon \cdot \dot{y}) \sum_{n=1}^{\infty} \frac{f^{(n)}(\phi)}{n!}(k \cdot y)^n \right]\right\} \tag{13}$$

where $S_c = \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \mathcal{L}_E(x_c, \dot{x}_c, \alpha_1)$ is the classical action corresponding to the extended Lagrangian $\mathcal{L}_E$. By introducing the functional differential operators

$$P_n(k, J) = \frac{1}{n!} \prod_{\ell=1}^{n} \left(\frac{k_{\nu_\ell}}{\mathrm{i}} \frac{\delta}{\delta J_{\nu_\ell}(\tau)}\right) \tag{14}$$

the propagator takes the form

$$K_{fi} = \mathrm{e}^{\mathrm{i}S_c} \exp\left\{\mathrm{i}\int_{\tau_i}^{\tau_f} \mathrm{d}\tau \left[-e(\epsilon_1 \cdot \dot{x}) \sum_{n=2}^{\infty} f^{(n)}(\phi) P_n(k, J) \right.\right.$$
$$\left.\left. - e\big(\epsilon_{1\nu_0}\big) \sum_{n=1}^{\infty} f^{(n)}(\phi) P_n(k, J) \frac{\mathrm{d}}{\mathrm{d}\tau} \frac{1}{\mathrm{i}} \frac{\delta}{\delta J_{\nu_0}(\tau)} \right]\right\} K[J]|_{J=0} \tag{15}$$

where

$$K[J] = \int_{(0,\tau_i)}^{(0,\tau_f)} \mathcal{D}[y(\tau)] \exp\left\{\mathrm{i}\int_{\tau_i}^{\tau_f} \mathrm{d}\tau \left[\frac{\dot{y}^2(\tau)}{4\alpha_1} + J_\mu(\tau) y^\mu(\tau) \right]\right\} \tag{16}$$

is the path integral for a free system in the presence of an external source $J^\mu(\tau)$. The latter is easily calculated and reads

$$K[J] = -\mathrm{i}(4\pi\alpha_1(\tau_f - \tau_i))^{-2} \exp\left\{ \frac{2\mathrm{i}\alpha_1 \Delta(J)}{\tau_f - \tau_i} \right\}. \tag{17}$$

Here $\Delta(J) = \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \int_{\tau_i}^{\tau} \mathrm{d}\sigma (\tau_f - \tau) J_\mu(\tau) J^\mu(\sigma)(\sigma - \tau_i)$ is the contribution of the source to the classical solution. By using the Green function of the classical free system, $G(\tau, \sigma) = [(\tau_f - \tau)(\sigma - \tau_i)\vartheta(\tau - \sigma) + (\tau_f - \sigma)(\tau - \tau_i)\vartheta(\sigma - \tau)]$, a simple computation shows that

$$\frac{\delta}{\delta J_{\nu_i}(\tau)} \mathrm{e}^{2\mathrm{i}\alpha_1 \Delta(J)/\Delta\tau} = (2\mathrm{i}\alpha_1/\Delta\tau)\, \mathrm{e}^{2\mathrm{i}\alpha_1 \Delta(J)/\Delta\tau} \int_{\tau_i}^{\tau_f} \mathrm{d}\sigma G(\tau, \sigma) J_{\nu_i}(\sigma) \tag{18}$$

and

$$\frac{\delta}{\delta J_{\nu_i}(\tau)} \frac{\delta}{\delta J_{\nu_j}(\tau)} \mathrm{e}^{2\mathrm{i}\alpha_1 \Delta(J)/\Delta\tau} = (2\mathrm{i}\alpha_1/\Delta\tau)\, \mathrm{e}^{2\mathrm{i}\alpha_1 \Delta(J)/\Delta\tau} g_{\nu_i\nu_j} G(\tau, \tau) + (4\mathrm{i}\alpha_1/\Delta\tau)\, \mathrm{e}^{2\mathrm{i}\alpha_1 \Delta(J)/\Delta\tau}$$
$$\times \int_{\tau_i}^{\tau_f} \mathrm{d}\sigma_1 \int_{\tau_i}^{\tau_f} \mathrm{d}\sigma_2 G(\tau, \sigma_1) J_{\nu_i}(\sigma_1) G(\tau, \sigma_2) J_{\nu_j}(\sigma_2). \tag{19}$$

Similar equations hold for the terms containing higher order derivatives.

We therefore see that both $\epsilon_{1\nu_0} P_n(k, J) \frac{\mathrm{d}}{\mathrm{d}\tau} \frac{1}{\mathrm{i}} \frac{\delta}{\delta J_{\nu_0}(\tau)} \exp\{2\mathrm{i}\alpha_1 \Delta(J)/\Delta\tau\}|_{J=0}$ and $P_n(k, J) \exp\{2\mathrm{i}\alpha_1 \Delta(J)/\Delta\tau\}|_{J=0}$, as well as the action of all possible mixed products of P_n corresponding to different values of the proper time over $\exp\{2\mathrm{i}\alpha_1 \Delta(J)/\Delta\tau\}$, are vanishing at $J^\mu = 0$ due to relations (7). In other words

$$\exp\left\{\mathrm{i}\int_{\tau_i}^{\tau_f} \mathrm{d}\tau \left[-e(\epsilon_1 \cdot \dot{x}) \sum_{n=2}^{\infty} f^{(n)}(\phi) P_n(k, J) \right]\right\} \exp\left\{\mathrm{i}\int_{\tau_i}^{\tau_f} \mathrm{d}\tau \left[-e \sum_{n=1}^{\infty} f^{(n)}(\phi) \right.\right.$$
$$\left.\left. \times P_n(k, J) \epsilon_{1\nu_0} \frac{\mathrm{d}}{\mathrm{d}\tau} \frac{1}{\mathrm{i}} \frac{\delta}{\delta J_{\nu_0}(\tau)} \right]\right\} K[J]|_{J=0} = K[J]|_{J=0} \tag{20}$$

and we finally get

$$K_{fi} = -\mathrm{i}(4\pi\alpha_1(\tau_f - \tau_i))^{-2}\, \mathrm{e}^{\mathrm{i}S_c} \tag{21}$$

so that the propagator is expressed in terms of the classical action only. To evaluate the action S_c we have therefore to solve the classical equations of motion (12) or by projecting on the basis $k^\mu, \tilde{k}^\mu, \epsilon_1^\mu, \epsilon_2^\mu$ the equations

$$\begin{aligned} &(k \cdot \ddot{x}) = 0 && (\epsilon_2 \cdot \ddot{x}) = 0 \\ &(\epsilon_1 \cdot \ddot{x}) = -2\alpha_1 e(k \cdot \dot{x}) f'(\phi) && (\tilde{k} \cdot \ddot{x}) = -4\alpha_1 e(\epsilon_1 \cdot \dot{x}) f'(\phi) \end{aligned} \tag{22}$$

where $f'(\phi)$ is the derivative with respect to the argument. Letting $\phi_i = k \cdot x_i$ and $\phi_f = k \cdot x_f$, their solution, although somewhat lengthy, is straightforward and reads as follows:

$$
\begin{aligned}
(k\cdot x)(\tau) &= (k\cdot x_i) + \frac{(k\cdot\Delta x)}{\Delta\tau}(\tau-\tau_i)\\
(\epsilon_1\cdot x)(\tau) &= (\epsilon_1\cdot x_i) + \left[\frac{(\epsilon_1\cdot\Delta x)}{\Delta\tau} + \frac{2\alpha_1 e}{(k\cdot\Delta x)}\int_{\phi_i}^{\phi_f}\mathrm{d}\phi f(\phi)\right](\tau-\tau_i) - \frac{2\alpha_1 e\Delta\tau}{(k\cdot\Delta x)}\int_{\phi_i}^{\phi}\mathrm{d}\phi f(\phi)\\
(\epsilon_2\cdot x)(\tau) &= (\epsilon_2\cdot x_i) + \frac{(\epsilon_2\cdot\Delta x)}{\Delta\tau}(\tau-\tau_i)\\
(\tilde{k}\cdot x) &= (\tilde{k}\cdot x_i) + \left[\frac{(\tilde{k}\cdot\Delta x)}{\Delta\tau} + \frac{4\alpha_1 e(\epsilon_1\cdot\Delta x)}{(k\cdot\Delta x)^2}\int_{\phi_i}^{\phi_f}\mathrm{d}\phi f(\phi) + \frac{8\alpha_1^2 e^2\Delta\tau}{(k\cdot\Delta x)^3}\left(\int_{\phi_i}^{\phi_f}\mathrm{d}\phi f(\phi)\right)^2\right.\\
&\quad\left. - \frac{4\alpha_1^2 e^2\Delta\tau}{(k\cdot\Delta x)^2}\int_{\phi_i}^{\phi_f}\mathrm{d}\phi f^2(\phi)\right](\tau-\tau_i) - \left[4\alpha_1 e\frac{\Delta\tau(\epsilon_1\cdot\Delta x)}{(k\cdot\Delta x)^2}\right.\\
&\quad\times\int_{\phi_i}^{\phi}\mathrm{d}\phi f(\phi) + 8\alpha_1^2 e^2\frac{(\Delta\tau)^2}{(k\cdot\Delta x)^3}\left(\int_{\phi_i}^{\phi_f}\mathrm{d}\phi f(\phi)\right)\int_{\phi_i}^{\phi}\mathrm{d}\phi f(\phi)\\
&\quad\left. - 4\alpha_1^2 e^2\frac{(\Delta\tau)^2}{(k\cdot\Delta x)^2}\int_{\phi_i}^{\phi}\mathrm{d}\phi f^2(\phi)\right]
\end{aligned}
\tag{23}
$$

where we have defined $\Delta x^\mu = \left(x_f^\mu - x_i^\mu\right)$.

The explicit form of the classical action turns out to be

$$
\begin{aligned}
S_c = \frac{\Delta\tau}{4\alpha_1}\left[\left(\frac{\Delta x}{\Delta\tau}\right)^2 + 4\alpha_1^2 m^2\right] &- \frac{\alpha_1 e^2\Delta\tau}{(k\cdot\Delta x)}\int_{\phi_i}^{\phi_f}\mathrm{d}\phi A^2(\phi) + \frac{\alpha_1 e^2\Delta\tau}{(k\cdot\Delta x)^2}\left(\int_{\phi_i}^{\phi_f}\mathrm{d}\phi A_\mu(\phi)\right)^2\\
&- \frac{e\Delta x^\mu}{(k\cdot\Delta x)}\int_{\phi_i}^{\phi_f}\mathrm{d}\phi A_\mu(\phi)
\end{aligned}
\tag{24}
$$

and the final integration of (21) over $\alpha_1\Delta\tau$, as in equation (11), gives the physical propagator in agreement with previous results [1, 4].

3. The path integral for the spinning particle

We now calculate the propagator for a spinning particle in an external plane wave. The pseudoclassical description of a spin-1/2 particle interacting with an arbitrary external electromagnetic field has been already described in [7] and the Lagrangian is

$$
\begin{aligned}
\mathcal{L}(x,\dot{x},\xi_\mu,\dot{\xi}_\mu,\xi_5,\dot{\xi}_5) &= -\frac{\mathrm{i}}{2}(\xi_\mu\dot{\xi}^\mu + \xi_5\dot{\xi}_5)\\
&\quad - \sqrt{m^2 - \mathrm{i}eF_{\mu\nu}\xi^\mu\xi^\nu}\sqrt{\left(\dot{x}^\mu - \frac{\mathrm{i}}{m}\xi^\mu\dot{\xi}_5\right)^2} - e\dot{x}_\mu A^\mu.
\end{aligned}
\tag{25}
$$

Analogous results, with minor differences, were found in [8], while the general theory of quantization Fermi–Bose systems is explained in [6].

The Lagrangian (25) is singular and produces the two first class constraints

$$\chi = (p-eA)^2 - m^2 + \mathrm{i}eF_{\mu\nu}\xi^\mu\xi^\nu \tag{26}$$

$$\chi_D = (p-eA)\xi - \mathrm{i}m\pi_5 - (m/2)\xi_5 \tag{27}$$

and the second class constraints

$$\chi_\mu = \pi_\mu - (\mathrm{i}/2)\xi_\mu. \tag{28}$$

The extended Hamiltonian, compatible with (26), (27) is

$$\mathcal{H}_E = \alpha_1[(p-eA)^2 - m^2 + \mathrm{i}eF_{\mu\nu}\xi^\mu\xi^\nu] + \mathrm{i}\alpha_2((p-eA)\xi) + m\alpha_2\left(\pi_5 - \frac{\mathrm{i}}{2}\xi_5\right). \tag{29}$$

The further second class constraints

$$\chi_5 = \pi_5 + (\mathrm{i}/2)\xi_5 \tag{30}$$

will be imposed directly on the states, thereby restricting the Hilbert space of the system. The relevant Dirac brackets are

$$\{\xi^\mu, \xi^\nu\} = \mathrm{i}\eta^{\mu\nu} \qquad \{\pi_5, \xi_5\} = -1. \tag{31}$$

According to the common practice for the path integration in quantum mechanics of an even number of Grassmann variables [14, 19], we substitute the ξ^μ variables with their holomorphic combinations

$$\eta_1 = \tfrac{1}{\sqrt{2}}(\xi^0 + \xi^3) \qquad \bar\eta_1 = -\tfrac{1}{\sqrt{2}}(\xi^0 - \xi^3) \tag{32}$$

$$\eta_2 = \tfrac{1}{\sqrt{2}}(\xi^1 + \mathrm{i}\xi^2) \qquad \bar\eta_2 = \tfrac{1}{\sqrt{2}}(\xi^1 - \mathrm{i}\xi^2) \tag{33}$$

(observe the useful identity $z_\mu\xi^\mu = -(\bar\eta_\alpha z_\alpha + \eta_\alpha \bar z_\alpha)$). For the path integration on the (ξ_5, π_5) variables we follow the procedure given in [13, 14]. The propagator takes the form

$$\begin{aligned} K_{fi} = &\int \frac{\mathrm{d}\pi_{5f}}{\mathrm{i}}\, \mathrm{e}^{\mathrm{i}\xi_{5f}\pi_{5f}} \int_{(\eta_i,x_i),\tau_i}^{(\bar\eta_f,x_f),\tau_f} \mathcal{D}(\eta,\bar\eta)\mathcal{D}(x) \int_{\xi_{5i},\tau_i}^{\pi_{5f},\tau_f} \mathcal{D}(\xi_5,\pi_5) \\ &\times \exp\left[\frac{\mathrm{i}}{2}(\pi_{5f}\xi_5(\tau_f) + \pi_5(\tau_i)\xi_{5i}) + \frac{1}{2}(\bar\eta_{\alpha f}\eta_\alpha(\tau_f) + \bar\eta_\alpha(\tau_i)\eta_{\alpha i})\right] \mathrm{e}^{\mathrm{i}S_E(\eta,\bar\eta,x)} \\ &\times \exp\left\{\mathrm{i}\int_{\tau_i}^{\tau_f}\left[\frac{1}{2}(\dot\pi_5\xi_5 + \dot\xi_5\pi_5) - m\alpha_2\left(\pi_5 - \frac{\mathrm{i}}{2}\xi_5\right)\right]\right\} \end{aligned} \tag{34}$$

where

$$S_E = \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \left\{\frac{\mathrm{i}}{2}(\bar\eta_\alpha\dot\eta_\alpha - \dot{\bar\eta}_\alpha\eta_\alpha) + \frac{1}{4\alpha_1}\dot x^2 + \alpha_1 m^2 + \frac{\mathrm{i}\alpha_2}{2\alpha_1}(\dot x\cdot\xi) - e(\dot x\cdot A) - \mathrm{i}e\alpha_1 F_{\mu\nu}\xi^\mu\xi^\nu\right\} \tag{35}$$

is the extended action. The shift from the classical or pseudoclassical path

$$x^\mu = x_c^\mu + y^\mu \qquad \eta_\alpha = \eta_{\alpha c} + \psi_a \qquad \bar\eta_a = \bar\eta_{\alpha c} + \bar\psi_\alpha \tag{36}$$

$$\xi_5 = \xi_{5c} + \psi \qquad \pi_5 = \pi_{5c} + \chi \tag{37}$$

implies that the boundary conditions

$$y^\mu(\tau_i) = y^\mu(\tau_f) = 0 \qquad \psi_\alpha(\tau_i) = \bar\psi_\alpha(\tau_f) = 0 \qquad \psi(\tau_i) = \chi(\tau_f) = 0 \tag{38}$$

have to be imposed when integrating over the shifted variables.

We next observe that the propagator factorizes as

$$K_{fi} = K_5 K_c K_q \tag{39}$$

where the meaning of the three factors will be explained here below.

First of all it is rather evident that K_5 refers to the contribution of the path integral over ξ_5 and π_5. The explicit calculation has been developed in [14] and we quote the final result

$$K_5 = (\xi_{5f} - \xi_{5i}) - m\alpha_2 \Delta\tau\, \mathrm{e}^{-\xi_{5f}\xi_{5i}/2}. \tag{40}$$

The factor K_c accounts for the classical contribution including the necessary surface terms, as explained in [19],

$$K_c(\eta_i, \bar{\eta}_f, x_i, x_f) = \exp\left\{\tfrac{1}{2}(\bar{\eta}_{\alpha f}\eta_{\alpha c}(\tau_f) + \bar{\eta}_{\alpha c}(\tau_i)\eta_{\alpha i}) + \mathrm{i}S_E(\eta_c, \bar{\eta}_c, x_c)\right\} \tag{41}$$

and the classical variables satisfy the equations of motion

$$\ddot{x}_c^\mu = -2e\alpha_1 F^\mu{}_\nu \dot{x}_c^\nu - \mathrm{i}\alpha_2 \dot{\xi}_c^\mu - 2\mathrm{i}e\alpha_1^2 \left(\frac{\partial}{\partial x_\mu} F_{\nu\rho}\right) \xi_c^\nu \xi_c^\rho \tag{42}$$

$$\dot{\xi}_c^\mu = -\frac{\alpha_2}{2\alpha_1}\dot{x}_c^\mu - 2e\alpha_1 F^\mu{}_\nu \xi_c^\nu. \tag{43}$$

The third factor K_q represents the contribution of the quantum fluctuations and it is a straightforward generalization of that obtained in the scalar case. We can assume, without losing in generality, that the plane-wave field is of the form

$$A^\mu(x) = -2^{-1/2}(\epsilon^\mu + \epsilon^{*\mu}) f(\phi) = \epsilon_1^\mu f(\phi) \tag{44}$$

with $\epsilon^\mu = \frac{1}{\sqrt{2}}(0, -1, \mathrm{i}, 0)$ and $\epsilon^{*\mu} = \frac{1}{\sqrt{2}}(0, -1, -\mathrm{i}, 0)$. The corresponding electromagnetic tensor is therefore

$$F^{\mu\nu}(x) = f^{\mu\nu} f'(\phi) \qquad f^{\mu\nu} = k^\mu \epsilon_1^\nu - k^\nu \epsilon_1^\mu \tag{45}$$

and we finally write

$$\begin{aligned} K_q = \int_{0,\tau_i}^{0,\tau_f} &\mathcal{D}[\psi_\alpha(\tau), \bar{\psi}_\alpha(\tau)]\mathcal{D}[y(\tau)] \exp\left\{\mathrm{i}\int_{\tau_i}^{\tau_f} \mathrm{d}\tau \left(\left[-\frac{\mathrm{i}}{2}(\psi\cdot\dot{\psi}) + \frac{1}{4\alpha_1}\dot{y}^2 + \mathrm{i}\frac{\alpha_2}{2\alpha_1}(\psi\cdot\dot{y})\right]\right.\right. \\ &- \left[e(\dot{x}\cdot\epsilon_1)\sum_{n=2}^\infty \frac{1}{n!} f^{(n)}(\phi)(k\cdot y)^n + e(\dot{y}\cdot\epsilon_1)\sum_{n=1}^\infty \frac{1}{n!} f^{(n)}(\phi)(k\cdot y)^n\right. \\ &+ \mathrm{i}e\alpha_1 \xi^\mu \xi^\nu f_{\mu\nu} \sum_{n=2}^\infty \frac{1}{n!} f^{(n+1)}(\phi)(k\cdot y)^n + \mathrm{i}e\alpha_1 \psi^\mu \psi^\nu f_{\mu\nu} \\ &\left.\left.\left.\times \sum_{n=0}^\infty \frac{1}{n!} f^{(n+1)}(\phi)(k\cdot y)^n + 2\mathrm{i}e\alpha_1 \xi^\mu \psi^\nu f_{\mu\nu} \sum_{n=1}^\infty \frac{1}{n!} f^{(n+1)}(\phi)(k\cdot y)^n\right]\right)\right\}. \end{aligned} \tag{46}$$

Recalling definition (14) of the functional differential operators and introducing Bose and Fermi external sources, J^μ and λ^μ respectively, we have

$$\begin{aligned} K_q = \exp&\left\{\mathrm{i}\int_{\tau_i}^{\tau_f} \mathrm{d}\tau \left[\mathrm{i}\frac{\alpha_2}{2\alpha_1}\frac{\delta}{\delta\lambda_\mu}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J^\mu}\right) - e(\dot{x}\cdot\epsilon_1)\sum_{n=2}^\infty f^{(n)}(\phi) P_n(k, J)\right.\right. \\ &- e\epsilon^{\nu_0}\sum_{n=1}^\infty f^{(n)}(\phi)\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J^{\nu_0}}\right) P_n(k, J) - \mathrm{i}e\alpha_1 f_{\mu\nu}\xi^\mu\xi^\nu \\ &\times \sum_{n=1}^\infty f^{(n+1)}(\phi) P_n(k, J) - \mathrm{i}e\alpha_1 f_{\mu\nu}\frac{\delta}{\delta\lambda_\mu}\frac{\delta}{\delta\lambda_\nu}\sum_{n=0}^\infty f^{(n+1)}(\phi) P_n(k, J) \\ &\left.\left.- 2\mathrm{i}e\alpha_1 f_{\mu\nu}\frac{\delta}{\delta\lambda_\mu}\xi^\nu \sum_{n=1}^\infty f^{(n+1)}(\phi) P_n(k, J)\right]\right\} K[J]G[\lambda]|_{(J=0,\lambda=0)}. \end{aligned} \tag{47}$$

Here $K[J]$ is given in (16), (17), while

$$G[\lambda] = \int_{0,\tau_i}^{0,\tau_f} \mathcal{D}(\psi_\alpha, \bar{\psi}_\alpha) \exp\left\{ i \int_{\tau_i}^{\tau_f} d\tau \left[-\frac{i}{2}\psi_\mu \dot{\psi}^\mu - i\lambda_\mu \psi^\mu \right]\right\}$$
$$= \exp\left\{ -\frac{1}{2}\int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} ds [\bar{\lambda}_\alpha(u) D(u,s) \lambda_\alpha(s)] \right\} \tag{48}$$

where

$$D(u,s) = 2\theta(\tau_f - u)\,\theta(u-s)\,\theta(s-\tau_i) \tag{49}$$

is the Green function of the free pseudoclassical system.

By taking into account the results obtained in equation (20) for the scalar case, the further relations

$$f_{\mu\nu}(\delta/\delta\lambda_\mu)(\delta/\delta\lambda_\nu) = -(\delta/\delta\bar{\lambda}_1)((\delta/\delta\lambda_2) + (\delta/\delta\bar{\lambda}_2)) \tag{50}$$

$$-(\delta/\delta\bar{\lambda}_1)((\delta/\delta\lambda_2) + (\delta/\delta\bar{\lambda}_2)) \exp\{-\bar{\lambda}_\alpha\lambda_\alpha\}|_{(\lambda=\bar{\lambda}=0)} = 0 \tag{51}$$

and

$$\left(\frac{\delta}{\delta\lambda^\mu} \frac{\delta}{\delta J_\mu} \xi^\rho f_{\rho\sigma} \frac{\delta}{\delta\lambda_\sigma} k^\nu \frac{\delta}{\delta J^\nu} \right) K[J]G[\lambda]|_{(J=\lambda=0)} = 0 \tag{52}$$

we finally have for K_q the same result as in the scalar case, namely

$$K_q = -i(4\pi\alpha_1\Delta\tau)^{-2}. \tag{53}$$

In conclusion, the propagator we are looking for is again fully determined once we compute the classical contribution K_c. We find it convenient to write the latter separating in the exponent the normalization factor of coherent states, the even Bose and even Fermi parts and the odd Fermi part, $\exp\{\bar{\eta}_{\alpha f}\eta_{\alpha i}\}$, E_B, E_F and O_F respectively, i.e.

$$K_c = \exp\{\bar{\eta}_{\alpha f}\eta_{\alpha i} + iE_B + E_F + \beta_2 O_F\} \qquad (\beta_2 = \alpha_2\Delta\tau). \tag{54}$$

$E_B E_F$ and O_F are obtained by solving the classical equations of motion. Explicitly, with $\beta_1 = \alpha_1\Delta\tau$,

$$E_B = \frac{(\Delta x)^2}{4\beta_1} + \beta_1 m^2 - \frac{e^2\beta_1}{(k\cdot\Delta x)}\int_{\phi_i}^{\phi_f} d\phi A^2(\phi) + \frac{e^2\beta_1}{(k\cdot\Delta x)^2}\left(\int_{\phi_i}^{\phi_f} d\phi A(\phi)\right)^2$$
$$- \frac{e\Delta x^\mu}{(k\cdot\Delta x)}\int_{\phi_i}^{\phi_f} d\phi A_\mu(\phi) \tag{55}$$

$$E_F = -\frac{2e\beta_1}{(k\cdot\Delta x)}\bar{\eta}_{1f}(\bar{\eta}_{2f} + \eta_{2i})(f(\phi_f) - f(\phi_i)) \tag{56}$$

$$O_F = \frac{1}{2\beta_1}[\bar{\eta}_{1f}(\tilde{k}\cdot\Delta x) + \bar{\eta}_{2f}(\epsilon^*\cdot\Delta x) - \eta_{1i}(k\cdot\Delta x) + \eta_{2i}(\epsilon\cdot\Delta x)]$$
$$- e(\bar{\eta}_{2f} + \eta_{2i})f(\phi_i) + \frac{e}{(k\cdot\Delta x)}(\bar{\eta}_{2f} + \eta_{2i})\int_{\phi_i}^{\phi_f} d\phi f(\phi)$$
$$+ \left[\frac{e\Delta x^\mu}{(k\cdot\Delta x)^2}\int_{\phi_i}^{\phi_f} d\phi A_\mu(\phi) + \frac{e^2\beta_1}{(k\cdot\Delta x)^2}\int_{\phi_i}^{\phi_f} d\phi A^2(\phi) \right.$$

$$- \frac{2e^2\beta_1}{(k\cdot\Delta x)^3}\left(\int_{\phi_i}^{\phi_f} d\phi A(\phi)\right)^2 - \frac{2e^2\beta_1}{(k\cdot\Delta x)^2}(f(\phi_f)+f(\phi_i))$$
$$\times \int_{\phi_i}^{\phi_f} d\phi f(\phi) + \frac{2e^2\beta_1}{(k\cdot\Delta x)} f(k\cdot x_f) f(k\cdot x_i)$$
$$\left. - \frac{e(\epsilon^*\cdot\Delta x)}{(k\cdot\Delta x)} f(k\cdot x_f) - \frac{e(\epsilon\cdot\Delta x)}{(k\cdot\Delta x)} f(k\cdot x_i)\right]\bar\eta_{1f}. \tag{57}$$

We now collect all the contributions K_5, K_q, K_c and integrate over the Lagrange multipliers associated with the first class constraints. We get an integrated kernel

$$K_{\text{int}} = \int_{-\infty}^0 d\beta_1 \int d\beta_2 \frac{-\mathrm{i}\,e^{\bar\eta_{\alpha f}\eta_{\alpha i}}}{(4\pi\beta_1)^2} e^{\mathrm{i}E_B} e^{E_F} e^{\beta_2 O_F}\left((\xi_{5f}-\xi_{5i}) - m\beta_2 e^{\frac{1}{2}\xi_{5f}\xi_{5i}}\right)$$
$$= \int_{-\infty}^0 d\beta_1 \int d\beta_2 \frac{-\mathrm{i}\,e^{\bar\eta_{\alpha f}\eta_{\alpha i}}}{(4\pi\beta_1)^2} e^{\mathrm{i}E_B}(1+E_F)\left((1+\beta_2 O_F)(\xi_{5f}-\xi_{5i}) - m\beta_2 e^{-\frac{1}{2}\xi_{5f}\xi_{5i}}\right). \tag{58}$$

This is not yet the true physical kernel. As already said, the second class constraint (30) imposes a restriction on the Hilbert space of the states. We will now produce the projection operator that does the job. Introducing, as in [14], the symbol # to indicate a change of sign for all the odd variables inside the state vector, we have

$$K_{\text{phys}} = \int \left(\psi_f^*(\xi_{5f})_{\text{phys}}\bar\Psi_f(\eta_f) K_{\text{int}} \psi_i(\bar\eta_i)^{\#}\psi_i(\xi_{5i})^{\#}_{\text{phys}}\right) d\mu(\eta_f)\, d\mu(\eta_i)\, d\xi_{5f}\, d\xi_{5i}$$
$$= \int 2^{-1/4}(1, 2^{-1/2})\begin{pmatrix}1\\ \xi_{5f}\end{pmatrix}\langle\psi_f|\gamma_0|\eta_f\rangle K_{\text{int}}\langle -\bar\eta_i|\psi_i\rangle 2^{-1/4}(1, -\xi_{5i})\begin{pmatrix}1\\ 2^{-1/2}\end{pmatrix}$$
$$\times d\mu(\eta_f)\, d\mu(\eta_i)\, d\xi_{5f}\, d\xi_{5i} \tag{59}$$

where $|\psi\rangle$ represents an arbitrary four-component spinor.

4. The kernel in spinor basis

The propagator contains the complete information on the quantum system and in particular the wave equation itself can be deduced from it. This is what we are going to do in this last section and to this purpose we begin by rewriting the different terms composing the physical kernel as matrix elements between states of the quantum space:

$$K_{\text{phys}} = \int_{-\infty}^0 d\beta_1 \frac{\mathrm{i}}{(4\pi\beta_1)^2} e^{\mathrm{i}E_B}\langle\psi_f|\gamma_0\left(\widehat{O}_F + \widehat{E_F O}_F + \frac{m\gamma_5}{\sqrt{2}}\widehat{E}_F + \frac{m\gamma_5}{\sqrt{2}}\right)|\psi_i\rangle. \tag{60}$$

The appearance of γ_5 in equation (60) is due to the fact that in the basis chosen for the coherent states $|\eta\rangle$ we have the relation $\gamma_5|\eta\rangle = -|-\eta\rangle$ [14]. It can be verified by a direct calculation that the explicit form of the operators reproducing the kernel (59) are the ones given here below,

$$\widehat{O}_F = -\frac{(\Delta x\cdot\widehat{\xi})}{2\beta_1} - \frac{e\Delta x^\mu}{(k\cdot\Delta x)^2}\int_{\phi_i}^{\phi_f} d\phi (k\cdot\widehat{\xi})A_\mu(\phi) + \frac{2e^2\beta_1}{(k\cdot\Delta x)^3}\left(\int_{\phi_i}^{\phi_f} d\phi A_\mu(\phi)\right)^2 (k\cdot\widehat{\xi})$$
$$- \frac{e^2\beta_1(k\cdot\widehat{\xi})}{(k\cdot\Delta x)}\left(\frac{1}{(k\cdot\Delta x)}\int_{\phi_i}^{\phi_f} d\phi (A^2(\phi) + (A(\phi_f)\cdot A(\phi)) + (A(\phi)\cdot A(\phi_i)))\right.$$

$$
- (A(\phi_f)\cdot A(\phi_i))\Bigg) + \frac{e}{(k\cdot\Delta x)}\int_{\phi_i}^{\phi_f} \mathrm{d}\phi\, (A(\phi)\cdot\widehat{\xi}) - e(A(\phi_i)\cdot\widehat{\xi})
$$

$$
+ \left(\frac{e(\epsilon^*\cdot\Delta x)}{(k\cdot\Delta x)} f(\phi_f) + \frac{e(\epsilon\cdot\Delta x)}{(k\cdot\Delta x)} f(\phi_i)\right)(k\cdot\widehat{\xi}) \tag{61}
$$

$$
\begin{aligned}
\widehat{E_F O_F} = -\frac{e}{(k\cdot\Delta x)}[&-(k\cdot\widehat{\xi})(\epsilon\cdot\widehat{\xi})(\epsilon^*\cdot\widehat{\xi})((\epsilon-\epsilon^*)\cdot\Delta x) \\
&+ (k\cdot\widehat{\xi})((\epsilon+\epsilon^*)\cdot\widehat{\xi})(\tilde{k}\cdot\widehat{\xi})(k\cdot\Delta x)](f(\phi_f) - f(\phi_i))
\end{aligned} \tag{62}
$$

and

$$
\widehat{E}_F = \frac{2e\beta_1}{(k\cdot\Delta x)}((k\cdot\widehat{\xi})([A(\phi_f) - A(\phi_i)]\cdot\widehat{\xi})). \tag{63}
$$

We must now choose a representation for the algebra of the operators $\widehat{\xi}^\mu$. Two possible choices are given by the following relations:

$$
\widehat{\xi}^\mu = \mathrm{i}\gamma^\mu/\sqrt{2} \qquad \widehat{\xi}^\mu = \gamma_5\gamma^\mu/\sqrt{2}. \tag{64}
$$

By substituting (64) into equations (61)–(63), the operator which represents the physical kernel becomes

$$
\begin{aligned}
\widehat{K}_{\mathrm{phys}}(f,\mathrm{i}) = \frac{\Gamma}{\sqrt{2}}\int_0^\infty \mathrm{d}s \frac{\mathrm{i}}{(4\pi \mathrm{s})^2}\exp\Bigg\{\mathrm{i}\Bigg[&-\frac{(\Delta x)^2}{4s} - m^2 s + \frac{e^2 s}{(k\cdot\Delta x)}\int_{\phi_i}^{\phi_f}\mathrm{d}\phi A^2(\phi) \\
&- \frac{e^2 s}{(k\cdot\Delta x)^2}\left(\int_{\phi_i}^{\phi_f}\mathrm{d}\phi A_\mu(\phi)\right)^2 - \frac{e\Delta x^\mu}{(k\cdot\Delta x)}\int_{\phi_i}^{\phi_f}\mathrm{d}\phi A_\mu(\phi)\Bigg]\Bigg\} \\
&\times\Bigg[m + \frac{\gamma\cdot\Delta x}{2s} + s\frac{em}{(k\cdot\Delta x)}(k\cdot\gamma)((A(\phi_f) - A(\phi_i))\cdot\gamma) \\
&+ \frac{e}{2(k\cdot\Delta x)}((k\cdot\gamma)(A(\phi_f)\cdot\gamma)(\Delta x\cdot\gamma) - (\Delta x\cdot\gamma)(k\cdot\gamma)(A(\phi_i)\cdot\gamma)) \\
&- \frac{e^2 s}{(k\cdot\Delta x)}(k\cdot\gamma)(A(\phi_f)\cdot\gamma)(A(\phi_i)\cdot\gamma) + \frac{e}{(k\cdot\Delta x)}\int_{\phi_i}^{\phi_f}\mathrm{d}\phi (A(\phi)\cdot\gamma) \\
&- \frac{e}{(k\cdot\Delta x)^2}(k\cdot\gamma)\Delta x^\mu\int_{\phi_i}^{\phi_f}\mathrm{d}\phi A_\mu(\phi) + \frac{e^2 s}{(k\cdot\Delta x)^2}(k\cdot\gamma)\int_{\phi_i}^{\phi_f}\mathrm{d}\phi A^2(\phi) \\
&- \frac{2e^2 s}{(k\cdot\Delta x)^3}(k\cdot\gamma)\left(\int_{\phi_i}^{\phi_f}\mathrm{d}\phi A_\mu(\phi)\right)^2 + \frac{e^2 s}{(k\cdot\Delta x)^2}(k\cdot\gamma) \\
&\times\int_{\phi_i}^{\phi_f}\mathrm{d}\phi((A(\phi_f)\cdot\gamma)(A(\phi)\cdot\gamma) + (A(\phi)\cdot\gamma)(A(\phi_i)\cdot\gamma))\Bigg]
\end{aligned} \tag{65}
$$

where Γ turns out to be a factor i for the first representation in equations (64) and $\Gamma = \gamma_5$ for the second one. For a different approach to this problem going in the same direction and leading to the same results see [20]. Note that if we choose the first and simpler representation we have to perform a Pauli–Gursey transformation on the physical spinors

$$
|\psi\rangle \to \exp\left\{\mathrm{i}\frac{\pi}{4}\gamma_5\right\}|\psi\rangle \tag{66}
$$

to obtain equation (65). In this case it is then easy to verify that

$$\left[\mathrm{i}\gamma^\mu \frac{\partial}{\partial x_f^\mu} - e\gamma^\mu A_\mu(x_f) + m \right] \Delta_F(x_f, x_i|A) = \mathrm{i}\sqrt{2}\widehat{K}_{\mathrm{phys}} \tag{67}$$

where we have defined

$$\begin{aligned} \Delta_F(x_f, x_i|A) = \int_0^\infty \mathrm{d}s \frac{-\mathrm{i}}{(4\pi s)^2} \exp\left\{ \mathrm{i}\left[-\frac{(\Delta x)^2}{4s} - m^2 s + \frac{e^2 s}{k\cdot\Delta x}\int_{\phi_i}^{\phi_f} \mathrm{d}\phi A^2(\phi) \right.\right. \\ - \frac{e^2 s}{(k\cdot\Delta x)^2}\left(\int_{\phi_i}^{\phi_f}\mathrm{d}\phi A_\mu(\phi)\right)^2 - \frac{e\Delta x^\mu}{(k\cdot\Delta x)}\int_{\phi_i}^{\phi_f}\mathrm{d}\phi A_\mu(\phi) \\ \left.\left. - \frac{es}{2(k\cdot\Delta x)}\int_{\phi_i}^{\phi_f}\mathrm{d}\phi\sigma^{\mu\nu}F_{\mu\nu}(\phi) \right]\right\} \end{aligned} \tag{68}$$

and $\Delta_F(x_f, x_i|A)$ satisfies the squared Dirac equation

$$\left[\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} + 2\mathrm{i}eA^\mu\frac{\partial}{\partial x^\mu} - e^2A^2 + m^2 + \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} \right]\Delta_F(x, y|A) = \mathrm{i}\delta^4(x-y). \tag{69}$$

We therefore conclude that the physical kernel is indeed the propagator for a spinor particle in the external electromagnetic field, since it satisfies the Dirac equation

$$\left[\mathrm{i}\gamma^\mu\frac{\partial}{\partial x^\mu} - e\gamma^\mu A_\mu(x) - m \right]\sqrt{2}\widehat{K}_{\mathrm{phys}}(x, y) = -\delta^4(x-y). \tag{70}$$

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